

D-DIMENSIONAL CONFORMAL σ -MODELS
AND THEIR TOPOLOGICAL EXCITATIONS

S.A.Bulgadaev

L.D.Landau Institute for Theoretical Physics
Moscow, RUSSIA

Max Plank Institute for Physics of Complex Systems
Dresden, GERMANY

A talk presented at the International Conference ICMP-2000,
17 - 22 July 2000, Imperial College, London, England

I. INTRODUCTION

The nonlinear σ -models (NSM) play an important role in the modern theoretical physics (see, for example [1, 2]). Usually, being an effective local theories, they describe the low-energy and long-wave behaviour of systems with degenerate vacuum manifolds \mathcal{M} .

The most interesting items in NSM are the existence and structure of the gapless modes spectra. The answers on these questions have some universality and depend only on dimension of space, symmetry and topology of \mathcal{M} .

The most important are the two-dimensional NSM, which have many interesting properties, including, at the classical level,

- 1) locality,
- 2) scale invariance,
and, on the quantum level,
- 3) renormalizability.

In some cases they have the next additional properties:

- 4) scale symmetry breaking and mass generation [1],
- 5) topological excitations (TE),
- 6) integrability (both on classical and quantum levels [3]).

The passage to other dimensions brings a lost of some of these properties. If, for example, one tries to conserve a locality this destroys a scale invariance. But, in some cases, a scale invariance is more important physical property than a locality. For this reason one needs to consider a scale invariant NSM in other dimensions. Such models can appear in the local scale-invariant systems with massless fluctuating fields, which induce an effective action for remaining fields. Usually, the gradient expansion method is used for obtaining of local effective action. However, in general, this method does not conserve a scale invariance of the initial systems. An exact effective action of this systems must be scale invariant and nonlocal. The last property is connected with long-range character of interactions induced by massless fluctuating fields. The well known example of such interaction in 3D space is the van der Waals interaction. Another important example is a $1/r^2$ potential in one-dimensional systems [4, 5, 6]. The corresponding NSM has many properties common with that of 2D models [7] and some applications to various physical systems [8].

The TE of the scale-invariant NSM also have the interesting properties. For example, the TE with logarithmically divergent energy induce in low-dimensional systems with $D \leq 2$ the topological phase transitions (TPT), which change drastically the correlations in these systems [9,10,11]. An existence of such TE depends on the nontriviality of the homotopic group $\pi_{D-1}(\mathcal{M})$.

Question :

Can such TE and TPT exist in NSM with $D > 2$?

Unfortunately, the TE with logarithmic energies do not appear in usual higher dimensional theories.

The main efforts in higher dimensional systems were devoted to the discovery of the TE with *finite energy* [12,13]. All such excitations give finite contribution to the partition function, but cannot induce a PT similar to the TPT, since the latter is induced by TE with *logarithmically divergent energy*.

Recently it was shown that the TE with logarithmic energy can exist in 3D conformal (or the van der Waals) NSM [14]. In this talk we introduce the conformal D-dimensional NSM and show that they can have different TE, including ones with logarithmical energies. An existence of last excitations is intimately related with a scale and conformal symmetries of the models.

II. D-DIMENSIONAL CONFORMAL σ -MODELS.

A condition of existence of TE with discrete topological charges and logarithmic energy puts over on NSM the following properties :

- 1) their homotopical group $\pi_{D-1}(\mathcal{M})$ must be nontrivial abelian discrete,
- 2) a scale invariance at classical level.

The first property permits some ambiguity in a dimension and a form of \mathcal{M} , while the second one defines a form of the NSM action \mathcal{S} almost uniquely in arbitrary dimensions.

A general expression for action \mathcal{S} of D -dimensional generalized NSM, admitting nonlocal ones, can be represented in the next form

$$\mathcal{S} = \frac{1}{2\alpha} \int d^D x d^D x' \psi^a(x) \boxtimes_{ab}^{(D)} (\psi|x, x') \psi^b(x'), \quad a, b = 1, \dots, n \quad (1)$$

where $\psi \in \mathcal{M}$, and n is its dimension. The form of the kernel \boxtimes depends on model. If the structures of the internal and physical spaces do not depend on each other and the latter space is homogeneous, then \boxtimes can be decomposed

$$\boxtimes_{ab}^{(D)} (\psi|x, x') = g_{ab}(\psi(x), \psi(x')) \square_D(x - x'), \quad (2)$$

where g_{ab} is some two-point metric function on \mathcal{M} . For local models an expression for \boxtimes can be defined in terms of manifold \mathcal{M} only

$$\boxtimes(\psi|x, x') = g_{ab}(\psi) \boxtimes \delta(x - x') \quad (3)$$

If the manifold \mathcal{M} can be embedded in Euclidean vector space $\mathbb{R}^{N(n)}$ with dimension $N(n)$, depending on n , then one can use instead of $g_{ab}(\psi, \psi')$ an usual Euclidean metric

$$g_{ab} = \delta_{ab}, \quad a, b = 1, \dots, N. \quad (4)$$

and the constraints defining \mathcal{M} . One must consider the manifolds with a discrete abelian homotopic group $\pi_{D-1}(\mathcal{M})$. The spheres S^{D-1} are the simplest among them with $\pi_{D-1}(\mathcal{M}) = \mathbb{Z}$. Then $N(D-1) = D$, and

$$\psi^a = n^a, \quad a = 1, \dots, D, \quad (\mathbf{n})^2 = 1. \quad (5)$$

where $\mathbf{n}(x)$ is a field of unit vectors in internal space \mathcal{R}^D . Since $N = D$ this internal space can be identified with a physical space. Below we confine ourselves by the simplest manifolds, the spheres. A possible generalizations will be shortly discussed in Section Y.

It is more convenient to write \mathcal{S} and \square_D in the momentum space

$$\mathcal{S} = \frac{1}{2\alpha} \int d^D x d^D x' (\mathbf{n}(x) \mathbf{n}(x')) \square_D(x - x') =$$

$$\frac{1}{2\alpha} \int \frac{d^D k}{(2\pi)^D} |n(k)|^2 \square_D(k) \quad (6)$$

For asymptotical scale invariance at large scales the kernel \square_D must have the next behaviour at small k

$$\square_D(k) \simeq |k|^D (1 + a_1(ka) + \dots) = |k|^D f(ka) \quad (7)$$

where a is a UV cut-off parameter, $f(ka)$ is some regularizing function with the next asymptotics

$$f(ka) = 1 + a_1 ka + \dots, \quad ka \rightarrow 0, \quad f(ka) \rightarrow 0, \quad ka \rightarrow \infty. \quad (8)$$

The kernel \square_D generalizes the usual Laplace kernel of two-dimensional σ -model

$$\square_2(k) \equiv \square(k) = -(\partial)^2(k) = k^2 \quad (9)$$

For this reason \square_D can be considered as a $|\partial|^D$ kernel. From (9) it follows that in even dimensional spaces \mathbb{R}^{2s} the kernel \square_{2s} is a local one

$$\square_{2s} = (-1)^s ((\partial)^2)^s. \quad (10)$$

\square_D is always a nonlocal one in odd dimensions $D = 2s + 1$ with a large-scale asymptotics

$$\square_D(x)|_{x \gg a} \simeq A_D / |x|^{2D}, \quad A_D = \frac{2^D \Gamma(D)}{\pi^{D/2} \Gamma(-\frac{D}{2})}. \quad (11)$$

Such kernels appear often in physics. The most known and important cases correspond to $D = 1$ [4, 5, 6] and to $D = 3$ [14].

In the higher dimensional spaces they can describe an effective, fluctuation induced, interactions. The similar nonlocal kernels exist in even-dimensional spaces too

$$\square_{2s}(x) \sim 1/x^{4s}, \quad (12)$$

but their Fourier-images contain an additional logarithmic factor

$$\square_{2s}(k) \sim k^{2s} \ln(k/k_0) \quad (13)$$

where k_0 is some UV cut-off parameter, regularizing a kernel (11) at small scales. This factor breaks some important properties of the model, for simplicity, we will discuss in even-dimensional spaces only a local kernels.

A conformal group in the higher-dimensional ($D > 2$) spaces is finite-dimensional [15]. Its main nontrivial transformation is an inversion transformation

$$x^i \rightarrow x^i/r^2, \quad r = |\mathbf{x}|. \quad (14)$$

The conformal invariance of \mathcal{S} with a kernel (11) follows from the next transformation properties of the kernel \square_D under conformal transformation (14) (the field $\mathbf{n}(x)$ and a coupling constant α are dimensionless)

$$x_i \rightarrow x'_i = x_i/r^2, \quad r \rightarrow r' = 1/r, \quad x_i/r = x'_i/r',$$

$$d^D x \rightarrow d^D x / |\mathbf{x}|^{2D}, \quad \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{2D}} \rightarrow \frac{|\mathbf{x}_1|^{2D} |\mathbf{x}_2|^{2D}}{|\mathbf{x}_1 - \mathbf{x}_2|^{2D}},$$

and, consequently,

$$\mathcal{S} = \frac{A_D}{2\alpha} \int d^D x_1 d^D x_2 \frac{(\mathbf{n}_1 \mathbf{n}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^{2D}} \quad (15)$$

is invariant and dimensionless. Thus the action (15) with a kernel (11) can be named a D-dimensional *conformal* NSM. Strictly speaking, a conformal invariance takes place only at large scales, since a kernel (11) needs some regularization at small distancies, which can break this invariance.

The corresponding Euler - Lagrange equation has a form

$$\int \square_D(x - x') \mathbf{n}(x') d^D x' - \mathbf{n}(x) \int (\mathbf{n}(x) \mathbf{n}(x')) \square_D(x - x') d^D x' = 0. \quad (16)$$

Its Green function $G^D(x)$ has the following form

$$G^D(x) = \square_D^{-1} = |\partial|^{-D} = \int \frac{d^D k}{(2\pi)^D} \frac{e^{i(\mathbf{k}\mathbf{x})}}{\square_D(k)} \quad (17)$$

At large scales $G^D(x)$ has a logarithmic asymptotic behaviour

$$G^D(x)|_{r \gg a} \simeq -B_D \ln(r/R), \quad B_D = \frac{1}{2^{D-1} \pi^{D/2} \Gamma(\frac{D}{2})}, \quad (18)$$

where R is a radius of the space or a size of system. Thus the conformal kernels \square_D coincide with kernels of the field theories equivalent to the generalized logarithmic gases [16].

III. TE WITH LOGARITHMIC ENERGY.

Since $\pi_{D-1}(S^{D-1}) = \mathbb{Z}$, there are the TE with topological charge $Q \in \mathbb{Z}$. The simplest TE with charge $Q = 1$, corresponding to the identical map of spheres S^2 , has the next asymptotic form (a "hedgehog")

$$n^i(x)|_{r \gg a} \simeq \frac{x^i}{r} \quad (19)$$

The action \mathcal{S} of this solution for odd D (with logarithmic accuracy) is

$$\mathcal{S} = \frac{\mathcal{C}_D}{\alpha} \ln(R/a), \quad \mathcal{C}_D = \frac{2^{D-2} \pi^{\frac{D-2}{2}} (D-1)^2 \Gamma^2(\frac{D-1}{2})}{\Gamma(\frac{D}{2})}. \quad (20)$$

The interaction of two different TE with charges Q_1 and Q_2 on large distancies has a form of the Green function $G^D(r)$

$$E_{12}(r) = Q_1 Q_2 G(r) \simeq -Q_1 Q_2 B_D \ln(r/a) \quad (21)$$

Due to "no hair-dressing theorem", only the "hedgehog" type solutions can exist in odd dimensional spaces [15]. In even dimensional spaces the transverse field configurations are also possible. The hedgehog energy for even D is

$$\mathcal{S} = \frac{\pi^{D/2} \Gamma(D+1)}{2\alpha \Gamma(D/2)} \ln(R/a), \quad (22)$$

In the usual local D-dimensional NS-model with action

$$\mathcal{S} = \frac{1}{2A} \int d^D x (\partial \mathbf{n})^2 \quad (23)$$

such TE have the energy

$$E \simeq \frac{S_{D-1}(D-1)}{2A(D-2)} (R^{D-2} - a^{D-2}). \quad (24)$$

For the mixed action \mathcal{S}_{mix} , containing a sum of kernels \square_d , $d = 2, D$, the corresponding equation has again the "hedgehog" solution (19) with total energy

$$E = \frac{S_{D-1}(D-1)}{2A(D-2)} (R^{D-2} - a^{D-2}) + \frac{\mathcal{C}_D}{\alpha} \ln(R/a). \quad (25)$$

where $S_{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ is a volume of the unit $(D-1)$ -dimensional sphere.

It means that a logarithmic part of the "hedgehog" energy in mixed models can be observed at scales

$$(A/\alpha)^{1/(D-2)} > l > a.$$

The analogous "anti-hedgehog" solutions with the same logarithmic energies exist also.

IV. OTHER TOPOLOGICAL EXCITATIONS.

The TE of instanton type with finite energy can also exist in the conformal NSM. They correspond to the configurations with trivial boundary condition

$$\mathbf{n}(x) \rightarrow \mathbf{n}_0, \quad r \rightarrow \infty, \quad (26)$$

where \mathbf{n}_0 is some constant unit vector. A necessary condition for their existence is a nontriviality of group $\pi_D(S^{D-1})$. Since

$$\pi_D(S^{D-1}) = \mathbb{Z}_2, \quad D > 3,$$

it means that the D-dimensional ($D > 3$) conformal NSM on spheres have the instanton-like TE only with \mathbb{Z}_2 topological charges

$$Q \in \pi_D(S^{D-1}) = \mathbb{Z}_2 = \mathbb{Z}(\text{mod}2).$$

In case $D = 3$ NSM has the TE with finite energies, the hopfions. They are characterized by topological charge, coinciding with the Hopf invariant $H \in \mathbb{Z}$ of the corresponding mapping $S^3 \rightarrow S^2$. This invariant is connected with linking number of two projected circles S^1

$$\{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{<\mathbf{r}_{12} \cdot [d\mathbf{r}_1 d\mathbf{r}_2]>}{|r_1 - r_2|^3}.$$

For one winding of one circle around another $H = 1$. If a mapping projects each circle q_i ($i = 1, 2$) times then $H = q_1 q_2$. This additional topological invariant

classifies "neutral" (relatively to group $\pi_2(S^2)$) configurations in all 3D NS-models defined on sphere S^2 .

Y. GENERALIZATIONS.

The natural generalizations are the conformal NSM on:

- 1) simple compact groups G ,
- 2) their homogeneous spaces G/H .

For G not all homotopic groups $\pi_i(G)$ are known. For classical groups all $\pi_i(G)$ are divided into stable homotopic groups and nonstable ones. The first ones correspond to

$$i \leq A_G n + B_G.$$

Here n is a rank of group G , A_G, B_G are some constants of order $O(1)$, depending on G . For groups $G = SO(n), U(n), Sp(n)$

$$A_{SO(n)} = 1, B_{SO(n)} = -2; \quad A_{U(n)} = 2, B_{U(n)} = -1;$$

$$A_{Sp(n)} = 4, B_{Sp(n)} = 1.$$

The stable homotopic groups do not depend on rank n and are denoted as $\pi_i(G)$. They are known for all classical groups and have a property of the Bott periodicity

$$\pi_i(G) = \pi_{i+b_G}(G)$$

where $b(G)$ is a corresponding period. For classical groups $G = U, SO, Sp$ the Bott periods have the next values [15]

$$b_U = 2, b_{SO} = 8, b_{Sp} = 4.$$

If one confines himself only by free (infinite) homotopic groups (without finite parts or torsion) then the nontrivial stable groups $\pi(G)$ correspond to the so-called characteristic classes of groups G [15]

$$\pi_k(G) \neq 0, \quad k = 2k_i(G) - 1, \quad 1 \leq i \leq n(G) \quad (37)$$

where $k_i(G)$ are the Weyl indices of group G . It follows from (38) that only odd homotopic groups can be nontrivial. All Weyl indices, the degree of the Weyl invariant polynomials on the maximal abelian Cartan subalgebra, are known [17]:

$$\begin{aligned} k_i(U(n)) &= 1, 2, \dots, n; \\ k_i(SO(2n+1)) &= 2, 4, \dots, 2n = k_i(Sp(n)); \\ k_i(SO(2n)) &= 2, 4, \dots, 2n-2; \\ (2n-2 \text{ appears twice for even } n \text{ and once for odd } n); \end{aligned}$$

There is a simplified form of the Bott periodicity for infinite homotopic groups of classical groups:

$$\begin{aligned} \pi_i(G) &= \pi_{i+4}(G) \quad \text{for } G = SO, Sp, \\ \pi_i(U) &= \pi_{i+2}(U). \end{aligned}$$

The nonstable groups are finite and are known only in some partial cases.

The TE with logarithmic energy and topological charges $Q \in \mathbb{Z}$ are possible only for even dimensional spaces with

$$D = 2k_i(G),$$

and the instanton-like TE - only for odd dimensional spaces with

$$D = 2k_i(G) - 1.$$

All above topological charges are scalars (i.e. one-component). In some cases (for example, for topological interpretation of the quantum numbers) it is very important to have TE with vectorial topological charges [18].

For conformal NS-models such TE can exist only in 3D space.

To obtain them one needs to consider NS-models on the maximal flag spaces $F_G = G/T_G$ of the simple compact groups G . They have

$$\pi_2(F_G) = \mathbb{L}_v,$$

where \mathbb{L}_v is a dual root lattice of G . This lattice, in general, is not enough for topological description of **all** quantum numbers of group G , (among them are $SU(n)$ groups). This fact can be connected with a problem of quark confinement [1, 18]. The 3D conformal NS-models on F_G have the TE with a logarithmic energy and interacting vector topological charges $\mathbf{Q} \in \mathbb{L}_v$. Since $\pi_3(F_G) = \pi_3(G) = \mathbb{Z}$, in this case the "neutral" configurations will also have different topological structures described by group $\pi_3(F_G)$.

For higher homotopic groups $\pi_i(F_G) = \pi_i(G)$, $i > 3$, thus in higher dimensions ($D > 3$) only scalar charges are possible in these models.

I would like to thank the organizers of the ICMP-2000 conference for the opportunity to present this talk and support.

This work was supported also by RFFI grant 00-15-96579.

References

- [1] Polyakov A.M., Gauge Fields and Strings, Harwood Academic Publishers, 1987.
- [2] Wegner F., Phys.Rev. **B19** (1979) 783. Efetov K, Larkin A.I., Khmel'nitskii D.E., ZETP **79** (1980) 1120; Levine H., Libby S.B., Pruisken A.M.M., Phys.Rev.Lett. **51** (1983) 1915;
- [3] Mikhailov A.V., Pis'ma v ZETP **23** (1976) 356; Mikhailov A.V., Zakharov V.E., ZhETP (1978) ?; Ogievetsky E., Wiegmann P.B., Phys.Lett. **B168** (1986) 360. Ogievetsky E., Reshetikhin N., Wiegmann P.B., Nucl.Phys. **B** (1986) ?.
- [4] Dyson F., Comm.Math.Phys. **12** (1969) 91, 212; **21** (1971) 269; Anderson P.W., Yuval G., Hamann D.R., Phys.Rev. **B1** (1970) 4464.

- [5] Calogero F., Jour.Math.Phys. **12** (1971) 419; Sutherland B.P., Jour.Math.Phys. **12** (1971) 246, 251;
- [6] Caldeira A.O., Leggett A.J., Phys.Rev.Lett. **46** (1981) 211; Ann.Phys. **149** (1983) 374;
- [7] Kosterlitz J.M. Phys.Rev.Lett. **37** (1976) 1577; Brezin E., Zinn-Justin J., Le Guillou J.C., J.Phys.**9** (1976) L119; Bulgadaev S.A., Phys.Lett. **A125** (1987) 299.
- [8] Korshunov S.E, Pis'ma v ZETP **45** (1987) 342; Bulgadaev S.A., Pis'ma v ZETP **45** (1987) 486; Schon G., Zaikin A.D., Phys.Rep. **C 198** (1990) 237.
- [9] Berezinsky V.L., JETP **59** (1970) 907; **61** (1971) 1545; Kosterlitz J.M., Thouless J.P., J.Phys. **C6** (1973) 118; Kosterlitz J.M., J.Phys. **C7** (1974) 1046.
- [10] Nelson D.R., Phys.Rev. **B18** (1978) 2318; Nelson D.R., Halperin B.I., Phys.Rev. **B19** (1979) 2457;
- [11] Bulgadaev S.A., Phys.Lett. **86A** (1981) 213; Nucl.Phys. **B224** (1983) 349; JETP (1999) ; hep-th/9906091.
- [12] t'Hooft G., Nucl.Phys., **B79** (1974) 276; Polyakov A.M., Pisma v ZETP, **20** (1974) 430.
- [13] Belavin A.A., Polyakov A.M., Schwartz A.S., Tyupkin Yu.S., Phys.Lett. **59B** (1975) 85.
- [14] Bulgadaev S.A., "3D van der Waals sigma-model and its topological excitations", hep-th/0002041; see also hep-th/9909023.
- [15] Dubrovin B.A., Novikov S.P., Fomenko A.T., Modern geometry, part I,II. Nauka, Moscow, 1979; part III. Nauka, Moscow, 1984.
- [16] Bulgadaev S.A., Phys.Lett., **87B** (1979) 47.
- [17] Bourbaki N., Groupes et algebres de Lie. Chapters IV-VI, Hermann, Paris, 1968.
- [18] Bulgadaev S.A., On topological interpretation of quantum numbers, hep-th/9901036, ZETP **116** (1999) 1131.